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A DIFFERENTIAL GAME OF SIMPLE APPROACH IN MANIFOLDS†

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A game situation is considered in which the players are two points of a manifold with a non-degenerate metric, each with controllable velocity. The payoff in the game is the minimum distance between the players in a semi-infinite interval of the time of motion. The first player minimizes the payoff, the second maximizes it. The phase space of the game is divided into subdomains. In one (the primary domain) the value of the game is the initial distance between the players, in the other (the secondary domain) it is less than the initial distance. It is shown that the boundary of the primary domain consists of singular optimal paths [1], and the regular paths approach it from both sides. Necessary conditions are established for the singular surface to be optimal and the equations of the singular paths are derived. They are of the same form as the analogous relationships in the game of pursuit [2].

A necessary optimality condition, formulated in terms of the geodesic distance between players, is found for the primary domain in the form of an inequality, enabling the boundary of the singular surface to be constructed. The existence of this boundary is a necessary condition for the secondary domain to be non-empty. A generalization of Bellman's equation is obtained; it is shown that the value of the game is constant along secondary optimal paths. On singular paths the distance between the players remains constant. The necessary conditions obtained here provide the basis for an algorithm for constructing optimal paths and the value of the game in the neighbourhood of singularities.

The algorithm is then used to work out a complete solution of the problem of approach on a two-dimensional cone, constructing the level curves of the value of the game and an optimal phase portrait. The set of cones for which the secondary domain is empty, i.e. for which the distance between the players is the value throughout the game space, is determined.

1. FORMULATION OF THE PROBLEM

IN AN n-DIMENSIONAL manifold N, we consider the motion of two points, P (the pursuer) and E (the evader), which is governed by the following relationships

$$\begin{gathered} P: x^{\cdot} = u, \quad u \in E_1(x), \qquad E: y^{\cdot} = v, \quad v \in E_\nu(y), \; \nu > 1 \\ E_\alpha(x) = \{ u \in R^n : |u|_x \leqslant \alpha \}, \quad |u|_x^2 = \langle G(x) u, u \rangle \end{gathered} \tag{1.1}$$

Here x and y are the local coordinate vectors of the points P and E, respectively, u and v are their velocities at the point x of N, and $G(x)$ is the metric tensor of the manifold—a non-singular positive definite matrix. Angle brackets denote the standard scalar product, i.e. the sum of products of the vector components.

In a previous paper, we considered a game of pursuit in which the player dynamics were similar to (1.1), but on the assumption that the pursuer enjoys superiority as to velocity, $\nu < 1$.

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Since the velocity of the pursuer in (1.1) is less than that of the evader, the point E may well evade l-capture, where l is the capture radius [4]. Moreover, in the Euclidean case, when $G(x)$ is the identity matrix throughout the space, player E may continuously increase the distance between P and E. In an arbitrary manifold, however, player P may succeed in reducing the initial distance between P and E.

In this connection, we will consider a positional differential game of points P and E with dynamics (1.1) over a semi-infinite time interval, in which player P tries to reduce the distance from himself to player E as much as possible, while the latter pursues the opposite objective.

Thus, the objective functional in our game problem is the quantity

$$J = \min_{0 \leqslant t < \infty} L(x(t), y(t)) \tag{1.2}$$

where L is the length of the minimum geodesic curve connecting the players. It is a solution of the following variational problem

$$\begin{gathered} L(x, y) = \min_{\xi(\cdot)} \int_{\sigma_0}^{\sigma_1} \sqrt{\langle G(\xi)\xi^{\cdot}, \xi^{\cdot}\rangle}\, d\sigma \\ \xi(\sigma_0) = x, \quad \xi(\sigma_1) = y \end{gathered} \tag{1.3}$$

where $\xi(\sigma)$, $\sigma_0 \leqslant \sigma \leqslant \sigma_1$ is a piecewise-smooth curve connecting the players. We will assume that the minimum (1.3) is attained in this class of curves. Using the first variation formula, we derive expressions (1.4) for the partial derivatives of the extremum value (a local minimum) of the functional (1.3) and corollaries (1.5) and (1.6) of those expressions, valid at points where $L(x, y)$ is differentiable

$$\begin{gathered} L_x(z) = G(x)a(z), \quad L_y(z) = G(y)b(z), \quad z = (x, y) \in R^{2n} \\ a = -\xi^{\cdot}(\sigma_0)/|\xi^{\cdot}(\sigma_0)|_x, \quad b = \xi^{\cdot}(\sigma_1)/|\xi^{\cdot}(\sigma_1)|_y \end{gathered} \tag{1.4}$$

where a and b are unit tangent vectors at the points P and E of the geodesic, directed toward the exterior of the curve PE; $z = (x, y)$ is the local coordinate vector in the phase space of the game $N \times N = N^2$.

Formulae (1.4) imply the eikonal equations

$$\langle G^{-1}(x)L_x, L_x\rangle = 1, \quad \langle G^{-1}(y)L_y, L_y\rangle = 1 \tag{1.5}$$

and the following extremum property of the players' motion along a geodesic

$$\begin{gathered} \min_u \max_v L^{\cdot} = \max_v \min_u L^{\cdot} = \nu - 1 \\ L^{\cdot}(x, y) = \langle L_x, u\rangle + \langle L_y, v\rangle \\ u^*(z) = -G^{-1}(x)L_x(z) = -a(z), \quad v^*(z) = \nu G^{-1}(y)L_y(z) = \nu b(z) \end{gathered} \tag{1.6}$$

Here and below extrema with respect to u and v are calculated with respect to the ellipsoids (1.2). The quantities u^* and v^* in (1.6) may be found by using the method of undetermined Lagrange multipliers.

We will assume that the manifold N is such that the global minimum (1.3) is attained on at most two curves. Let Γ_0 be the set of boundary values $z = (x, y)$ in (1.3) for which two minima exist. Let us assume that a non-empty neighbourhood of Γ_0 exists in which smooth functions $L^+(z)$, $L^-(z)$ are defined and the global minimum can be written in the form

$$L(z) = \min\,[L^+(z), L^-(z)] \tag{1.7}$$

2. THE STRUCTURE OF THE SOLUTION AND THE NECESSARY OPTIMALITY CONDITIONS

We will assume that the value $V(z)$ of the game (1.1), (1.2) exists and is continuous and directionally differentiable. Note the obvious inequality $V(z) \leqslant L(z)$, $z \in Z$, which also follows from (1.3) and means that the least distance between the players cannot exceed the initial distance. Here Z is the domain in some system of local coordinates of N^2. The subdomain $Z_1 \subset Z$ in which $V(z) = L(z)$ will be called the primary domain, and the subdomain $Z_2 = Z \setminus Z_1$ the secondary domain. If $z \in Z_2$, then $V(z) < L(z)$.

Thus, by definition, player P, starting from a point of the secondary domain, can reduce the initial distance between the players.

The function L as expressed in (1.7) is differentiable with respect to the direction $w = (u, v)$. At points of smoothness the directional derivative $\partial L / \partial w$ is identical with the total derivative with respect to time: $\partial L / \partial w = L^{\cdot}$.

Lemma 1. In the primary domain Z_1, the following condition holds

$$\min_u \max_v \partial L/\partial w \geqslant 0 \tag{2.1}$$

Indeed, if this were false, there would be a positional control for player P making $\partial L / \partial w < 0$. This means that the distance L_Δ would be less than the initial distance after a sufficiently short time interval $\Delta > 0$, contrary to the definition of the primary domain.

Lemma 2. The following generalized necessary conditions for optimality hold for the value of the game $V(z)$ secondary domain

$$\min_u \max_v \partial V/\partial w \geqslant 0 \geqslant \max_v \min_u \partial V/\partial w \tag{2.2}$$

Inequalities (2.2) may be verified by indirect reasoning: the reverse of the left (right) inequality in (2.2) would imply that player P (E) could achieve an outcome superior to the value of the game [3].

At points of smoothness, inequalities (2.2) become equalities and imply Bellman's equation

$$F(z, p) = -\sqrt{\langle G^{-1}(x) V_x, V_x \rangle} + \nu \sqrt{\langle G^{-1}(y) V_y, V_y \rangle} = 0 \tag{2.3}$$

which means that $dV/dt = 0$ along an optimal path, i.e. the value of the game is constant along optimal paths. This also follows from the form of the functional (1.3) and the definition of the secondary domain. Indeed, if $z(t)$ is a path starting at a point $z^0 \in Z_2$, then $L(z(t))^{\cdot} \geqslant L(z(t_*))$, where t_* is either the first time the point hits the boundary ∂Z_2, or $t_* = \infty$ if the path never leaves Z_2. By the definition of t_*, we have $z(t) \in Z_2$, $0 \leqslant t < t_*$. If the above inequality is false, there is a time $t_1 < t_*$ such that $L(t) \leqslant L(t_*)$ for $t_1 \leqslant t \leqslant t_*$. This means that the value of the game at $z = z(t_1)$ is the initial distance $L(t_1)$, i.e. $z(t_1) \in Z_1$, contrary to the condition $z(t_1) \in Z_2$.

Thus, the value of the game in the secondary domain is equal to the distance $L(t_*)$, which is reached at time $t_* = \infty$ if the path never leaves Z_2, or at the moment the boundary Γ of Z_2 is reached: $\Gamma = \partial Z_2$.

Note that, unlike the pursuit problem considered in [3], a primary solution in the approach problem does not satisfy Bellman's equation for the secondary domain. Indeed, by the eikonal equations (1.5), substitution of a primary solution into the function (2.3) yields the following equation for $L(z)$ and a corresponding expression for the primary velocity

$$F(z, L_z(z)) = \nu - 1, \quad z^{\cdot} = F_p(z, L_z(z)) \tag{2.4}$$

Bellman's equation turns out to be true only when $\nu = 1$. In the domain in which $V(z)$ is twice smooth, the players' optimal paths are defined by the characteristic system of equations (2.3)

$$z^{\cdot} = F_p, \quad p^{\cdot} = -F_z \tag{2.5}$$

If a pair $z(t)$ and $p(t)$ is a solution of system (2.5), $z(t)=(x(t), y(t))$, then one can show that both functions $x(t)$ and $y(t)$ are extremals of problem (1.3). Thus, in regular motion the players move along a geodesic—generally a different geodesic for each player.

3. THE BOUNDARY OF THE SINGULAR SURFACE

Our previous discussions show that optimal paths may cross over from the secondary to the primary domain. The optimal behaviour of the players in the primary domain is not necessarily unique. In this situation player P plays a passive role, since the optimal outcome of the game—the distance between the players—is fixed from the start, and the future behaviour of P makes no difference. On the other hand, player E must have at his disposal some way to guarantee that the initial outcome should not become less favourable. This might be, say, an open-loop control of E over a semi-infinite time interval, or a positional control—a field of velocities guaranteeing local non-decrease of the distance. At points where $L(z)$ is differentiable such a field is defined, for example, by the primary control $v^*(z)$ in (1.6). At points where L is not smooth a favourable control for E may require information about the instantaneous value of the control of P, or the outcome may turn out to be attainable with ϵ-accuracy.

We will use Lemma 1 to determine the necessary conditions that define the surface Γ in an optimal synthesis.

The assumptions of Lemma 1 are trivially satisfied at points where L is smooth, by property (1.6). The function (1.7) is not smooth on the surface $\Gamma_0=\{z\in Z,\ L^+(z)=L^-(z)\}$.

Let us determine what part of Γ_0 lies in Z_1. We will use condition (2.1). On Γ_0 we have $\partial L/\partial w=\min[L^{+\bullet}, L^{-\bullet}]$, $L^{\pm\bullet}=\langle L_x^{\pm}, u\rangle+\langle L_y^{\pm}, v\rangle$, $w=(u, v)$. Calculations similar to those in the proof of the theorem of [3] give

$$\min_u \max_v \min_{\pm} L^{\pm} = \min[0, F(z, R_z(z))]$$
$$R(z) = (L^+(z)+L^-(z))/2$$

(for the definition of F, see (2.3)).

Thus, by Lemma 1, Z_1 contains only the part Γ_1 of Γ_0 defined by the condition $F(z, R_z(z)\geqslant 0$. Denote the boundary of Γ_1 by B. Using (1.1), (1.4) and (2.3), express the formula $F(z, R_z(z)=0$ in terms of the norms of the vectors a^++a^-, b^++b^-. Then the manifold B is defined by two equalities

$$B: L^+(z)=L^-(z),\quad |a^++a^-|_x=\nu|b^++b^-|_y \tag{3.1}$$

Thus, the set B is also a part of the required boundary surface, $B\subset\Gamma$. To construct Γ, we can use the method of singular characteristics [3], first finding the Cauchy data in the boundary manifold B. To apply the technique of [3], we need three necessary optimality conditions in the form of equalities in terms of a point $z\in\Gamma$ and a corresponding quantity p—the limiting value of the gradient of V in the secondary domain.

As the first two conditions we use Bellman's equation (2.3) and the continuity of the value of the game: $V(z)=L(z)$, $z\in\Gamma$. The third condition may be obtained by assuming that Γ consists of singular optimal paths. Then, due to the uniqueness of the extrema (2.2), a tangency condition $\langle F_p, p-q\rangle=0$, $p=Vz$, $q=L_z$ will hold at the points of Γ, that is, the paths leaving Z_2 reach Γ as tangents (Fig. 1). The tangency condition may also be verified independently by varying the surface Γ, as was done in the proof of the theorem in [4]. To that end we need only assume that paths reaching Γ from Z_2 do so in a finite time (the time t_* in Sec. 2 is finite).

Thus, we shall assume that paths from Z_2 reach the surface in a finite time. Then the paths reach the surface as tangents, and the following three optimality conditions hold on Γ

$$F_0(z, p)=0,\quad F_1(z, V)=L-V=0,\quad F_{-1}(z, p)=\langle F_p, p-q\rangle=0 \tag{3.2}$$

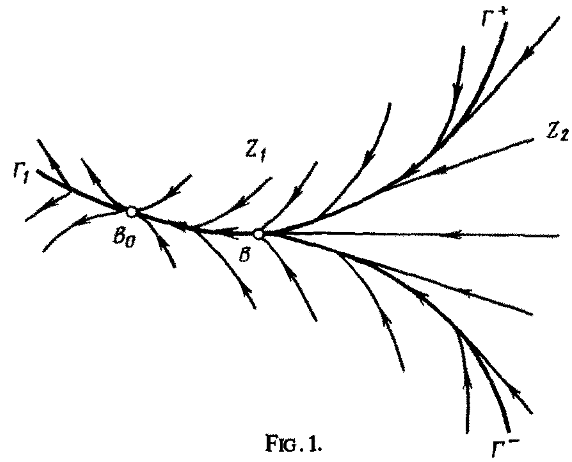

FIG. 1.

Here p is the limiting value of the gradient of V in the secondary domain, $q=L_z=(L_x, L_y)$. The first equation in (3.2) is Bellman's equation, the second is the continuity condition for the value, and the third is the condition that regular paths from the secondary domain are tangent to the singular surface.

According to the technique of [3], the equations of the singular characteristics are written in terms of the Hamiltonian and the corresponding characteristic system

$$\mu H = \{F_{-1}F_1\}F_0 + \{F_1F_0\}F_{-1} + \{F_0F_{-1}\}F_1$$
$$\dot z = H_p, \quad \dot p = -H_z - H_V p, \quad \dot V = \langle p, H_p\rangle$$

Here μ is the homogeneity multiplier and $\{F, G\}=\langle F_z+pF_V, G_p\rangle-\langle G_z+pG_V, F_p\rangle$ are the Jacobi brackets.

The characteristic system for a function F_i of the type (3.2) may be written, in view of the identity $F_{-1}\equiv\{F_1F_0\}$, in the form

$$\dot z = F_{0p}, \quad \dot p = -F_{0z} - \{F_0\{F_1F_0\}\}/\{F_1\}\{F_0F_1\}\}\ (p-q) \tag{3.3}$$

These equations can be used to construct the singular surface if the corresponding value of the vector p is known for $z\in B$.

Differentiating the equality $V(z)-L(z)=0$ with respect to the $2n-2$ tangent directions of B, we obtain, besides the first and third equalities of (3.2), a system of equations for p which has the following solution [3, Lemma 5]

$$p(z) = R_z(z) = (q^+(z)+q^-(z))/2, \quad z\in B, \quad (q^{\pm}=L_z^{\pm}) \tag{3.4}$$

To construct the singular surface $\Gamma=\partial Z_2$, we must integrate (3.3) in inverse time with initial values $z=z^0$, $p=(q^+(z^0)+q^-(z^0))/2$, $z^0\in B$.

If G is the identity matrix in some domain, the function F in (2.3) for that domain will have the form

$$F(p) = -\sqrt{\sum_{i=1}^{n} p_i^2} + v\sqrt{\sum_{i=n+1}^{2n} p_i^2} \tag{3.5}$$

System (3.3) is simplified

$$\dot z = F_p, \quad \dot p = [\langle S_{zz}F_p, F\rangle/\langle F_{pp}q, q\rangle]\ (p-q) \tag{3.6}$$

We note some properties of primary paths in the neighbourhood of B. The conditions for the primary paths, i.e. integral curves of system (2.4), to touch the surface Γ_0 on both sides are

$$\langle F_p(z, q^+), q^+-q^-\rangle=0, \quad \langle F_p(z, q^-), q^+-q^-\rangle=0, \quad q^{\pm}=L_z^{\pm} \tag{3.7}$$

A check shows that both equalities hold simultaneously in some set B_0 which, unlike the situation in the pursuit problem in [3], is a subset of the primary domain, $B_0 \subset \Gamma_1$. On the part of Γ_1 between B and B_0, the primary paths approach from both sides. The other part of Γ_1, as in the pursuit problem, is a dispersal surface, i.e. the primary paths leave it in both directions (Fig. 1). Note that if $\nu = 1$ the sets B and B_0 are identical. In that case, the above-mentioned system of equations for p has, besides the solution (3.4), a one-parameter family of solutions

$$p_\lambda = \frac{1+\lambda}{2} q^+ + \frac{1-\lambda}{2} q^- \tag{3.8}$$

This means that the gradient of $V(z)$ cannot be extended continuously to the set B, which may be approached by a one-parameter family of solutions of system (2.5)—an integral funnel.

The paths from the primary domain approach the singular surface Γ and never leave it, as was the case for the equivocal surface in the pursuit problem. Thus, qualitatively speaking, the surface Γ and the part of Γ_1 between B_0 and B recall a universal surface as defined in [1].

These assertions may be proved along lines analogous to those in [3], using the relationships (2.3), (2.4), (3.2) and (3.4).

Based on the considerations of this section, we can propose a four-stage algorithm that will construct optimum paths and compute the value of the game in the neighbourhood of singular surfaces.

In the first stage, the variational problem (1.3) is solved, to construct the functions $L^+(z)$ and $L^-(z)$ in a domain which may well be larger than Z_1. This yields the primary solution.

In the second stage, the relationships (3.1) and (1.4) are used to construct the set B. If it is empty, the analysis of the problem is complete, and the primary solution is a solution of the game in the whole space. If B is not empty, one goes on to the third stage.

In the third stage, the branches Γ^+ and Γ^- of the enveloping singular surface are constructed. To that end one integrates system (3.3) in retrograde time with initial data $z = z^0$, $p = R_z(z^0)$, $z^0 \in B$, using the functions L^+ and q^+ (L^- and q^-) for the branch Γ^+ (Γ^-). The surface Γ separates the domains Z_1 and Z_2.

In the fourth stage, the domain Z_2 is filled up by optimal paths by integrating system (2.5) in retrograde time with initial data $z = z^0$, $p = p(z^0)$, $z^0 \in \Gamma$.

4. THE PROBLEM OF APPROACH ON A TWO-DIMENSIONAL CONE

We will now let N be a convex conical surface $K + O$ (K is the cone without its apex O) in Euclidean 3-space; the metric on N is that of the latter. To describe the players' movements on the cone, we introduce a Euclidean local system of coordinates as follows. Fold up the cone along an arbitrary pair of opposing generators γ^+, $\gamma^- \subset K$ into a plane two-sided angle of magnitude α, $0 \leqslant \alpha \leqslant \pi$. In the plane of this angle, take a Cartesian (rectangular) system of coordinates with origin at the apex of the cone and abscissa axis along the bisector (Fig. 2).

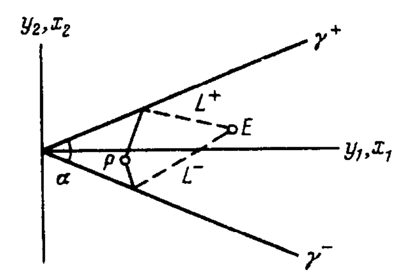

FIG. 2.

The coordinates $x=(x_1,\ x_2)$, $y=(y_1,\ y_2)$ of points P and E yield a Euclidean system of local coordinates in which the equations of motion of these points are

$$P: x^{\cdot}=u,\ |u| = \sqrt{u_1^2+u_2^2}\leqslant 1;\ E: y^{\cdot}=v,\ |v|\leqslant \nu, \nu>1 \tag{4.1}$$

For a detailed derivation of these and some of the other formulae in this section we refer the reader to [3]. The above transformation of the cone does not change the lengths of geodesics. We may thus interpret a game of approach on the cone as an equivalent game on the angle. The geodesics on the angle that yield the local minima $L^{\pm}$ of (1.7) form two families of two-armed polygonal lines, of lengths

$$L^{\pm}(z)=[|x|^2+|y|^2-2(x_1y_1-x_2y_2)\cos\alpha \mp 2(x_2y_1+x_1y_2)\sin\alpha]^{1/2} \tag{4.2}$$

where $(x_1,\ x_2)$, $(y_1,\ y_2)$ are the local Euclidean coordinates of P and E, respectively. The functions (4.2) define a solution to the first stage of the algorithm in Sec. 3. They are obtained on the assumption that P and E lie on different sheets of the angle; this may be ensured by suitable choice of the generators $\gamma^{\pm}$.

We now introduce self-similar variables ρ, φ and τ, related to the original variables as follows: $\rho=r/R$, $dt/d\tau=R$, where r and R are the distances of E and P respectively, from the apex, and φ is the angle between them on the flattened surface of the cone. Note that the self-similar time τ is related to the initial non-integrable differential relation. The equations of motion and the objective functional may be written in terms of the self-similar variables [3] as follows:

$$\rho'=v_1-\rho u_1,\ \varphi'=v_2/\rho-u_2,\ |u|\leqslant 1,\ |v|\leqslant \nu$$

$$J=\min_{0\leqslant\tau<\infty}\sqrt{1+\rho^2-2\rho\cos\varphi}\,\exp\left(\int_0^{\tau}u_1(\xi)\,d\xi\right)$$

The quantity to be minimized in the functional J is equal to the quotient L/R at $\tau=0$.

The self-similar variables may be interpreted as follows. Equip the players with complex coordinates z_P and z_E in the flattened surface of the cone: $z_P=z_P^0e^{i\varphi_P}$, $z_E=z_E^0e^{i\varphi_E}$. Then the quotient $w=z_E/z_P=\rho e^{i(\varphi_E-\varphi_P)}$ is a complex number whose modulus and argument are precisely the self-similar variables ρ, φ.

In self-similar variables the set B becomes a point with coordinates $\rho=\rho_B$, $\varphi=\alpha$. An equation for ρ_B may be derived by using (4.2) and (3.1) To do this we observe that, because of the rotational symmetry of the cone, any point of the manifold may be put in the form $(x_1, 0, y_1, 0)$. Suitable choice of the unit of measurement will equate one of the parameters x_1 or y_1 to any specified number, say, $y_1=1$. Then $x_1=1/\rho_B$, and the equation for ρ_B will be

$$-|1-\rho\cos\alpha|+\nu|\rho-\cos\alpha|=0 \tag{4.3}$$

Equation (4.3) has no roots in the domain $\Pi_1=\{(\alpha,\ \nu),\ \pi/2<\alpha\leqslant\pi,\ \nu>-1/\cos\alpha\}$ of parameter values; its root in the domain $\Pi_2=\Pi\backslash\Pi_1$, $\Pi=\{(\alpha,\ \nu):0\leqslant\alpha\leqslant\pi,\ \nu>-1\}$, however, is not unique. An analysis based on the necessary optimality conditions shows that in the optimum construction the domain Π_2 can contain only one root

$$\rho_B=(1+\nu\cos\alpha)/(\nu+\cos\alpha),\ (\alpha,\nu)\in\Pi_2 \tag{4.4}$$

On the separating curve $\Pi_*=\{(\alpha,\ \nu),\ \alpha\geqslant\pi/2,\ \nu>-1/\cos\alpha\}$ we have $\rho_B=0$.

The geometrical meaning of the solution (4.4) is as follows. Consider the players' motion along the opposing generators γ^+ and γ^-, with P moving toward the apex and E away from it. The players' distances from the apex are linear functions of time: $R=R_0-t$, $r=r_0+\nu t$. In this motion we have $L^+(z)=L^-(z)=(r^2+R^2-2rR\cos\alpha)^{1/2}$. Substituting the functions $r(t)$ and $R(t)$

into $L(r, R)$ and minimizing L as a function of time, we see that $r(t^*)/R(t^*) = \rho_B$ at the point of minimum $t = t^*$.

For the domain Π_1 in which Eq. (4.3) has no roots, the domain Z_2 is empty, the value of the game is the initial distance between the players, and the algorithm of Sec. 3 is completed.

For the domain Π_2, we must carry out the third and fourth stages of the algorithm. We first integrate the system of singular characteristics (3.6) with functions (3.5) $(n=2)$ and (4.2), assuming standard initial data $z_B = (1/\rho_B, 0, 1, 0)$ and $p_B = \frac{1}{2}(q_B^+ + q_B^-)$. Due to the symmetry of the manifold, all the other paths that make up Γ are obtained by simple calculation from the standard ones (see [4]). Then, to fill out the domain Z_2, we proceed in retrograde time from the points of the surface Γ, solving the first-order system (2.5) with the function (3.5)

Figure 3, drawn in polar coordinates ρ and φ, shows a symmetric half of the synthesis pattern for Π_2. The curve BD represents the singular surface Γ. The point $D(\rho = \rho_0,\ \varphi = 0)$ is reached in a finite time, during which the distance between the players remains constant. If player E starts at some point of BD along the geodesic BD, the distance between the players begins to increase. However, the path will then go off into the secondary domain and the final outcome for player E will be inferior. We have simulated such situations numerically.

The segment OD is a dispersal curve, at whose endpoints the generalized necessary conditions for optimality (2.2) are satisfied.

Figure 3 shows the primary paths corresponding to motion of the players along the geodesics connecting them. These paths either approach the singular surface Γ or go to the point $\rho = \nu$, $\varphi = 0$ in infinite time. The primary motions begin in a neighbourhood of the point $\rho = 1$, $\varphi = 0$ or on the dispersal part of the ray $\varphi = \alpha$, $\rho \geqslant \sqrt{\nu}$. Using the tangency conditions (3.7), one can show that a primary path will touch the ray $\varphi = \alpha$ at $\rho = \sqrt{\nu}$.

The primary paths may be constructed as a mapping $z \to \rho$, φ of the integral curves of system (2.4). The parametric representation of these paths in polar coordinates ρ and φ is given by formulae (2.11) of [3], with the sole difference that the sign of tg φ is reversed. Using the complex interpretation of the self-similar variables, one can show that the images of all regular paths are circles in the p, φ plane. Indeed, the primary and secondary regular paths of the players in the complex plane of the flattened cone lie along the straight lines $z_P(t) = z_P^0 + u^0 t$, $z_E = z_E^0 + v^0 t$, where z_E^0, z_P^0, u^0, v^0 are the complex coordinates and velocities of the players and t is real time. This follows from the remark at the end of Sec. 2. For the homogeneous complex variable introduced previously, we have

$$w = (z_E^0 + v^0 t)/(z_P^0 + u^0 t)$$

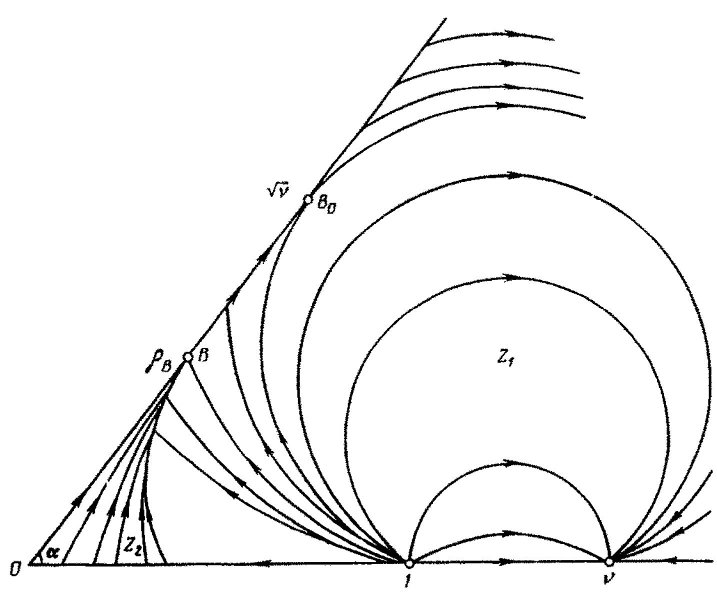

FIG. 3.

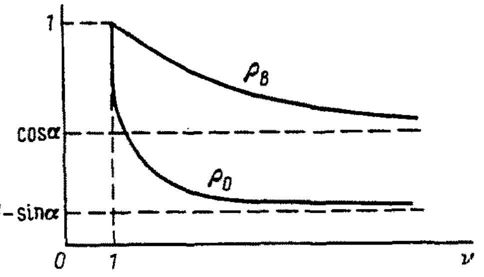

FIG. 4.

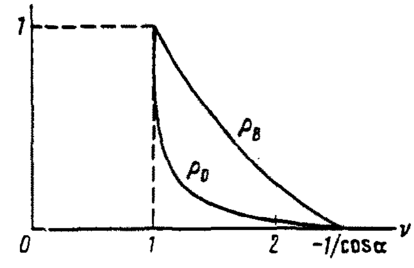

FIG. 5.

This Möbius transformation maps the real axis of the z-plane into a circle in the w-plane. In the case of primary paths, the centres of these circles lie in the straight line perpendicular to the axis $\varphi = 0$ and passing through the point $\rho = (1+\nu)/2$.

The functions $\rho_B(\nu)$ and $\rho_0(\nu)$, plotted numerically for fixed α, are shown in Fig. 4 for $\alpha = 2\operatorname{arctg}\frac{1}{2}$ in the interval $[0, \pi/2]$; Fig. 5 is a similar construction for $\alpha = 2\operatorname{arctg}\frac{3}{2}$ in the interval $[\pi/2, \pi]$. Let $\rho_B{}^*$, $\rho_0{}^*$ denote the limits of ρ_B and ρ_0 as $\nu \to \infty$ for $\alpha < \pi/2$. As is evident from formula (4.4), $\rho_B{}^* = \cos\alpha$. We will show that $\rho_0{}^* = 1 - \sin\alpha$. Consider the initial position of the players, represented by the point D in Fig. 3. If ν is large enough, while the phase point moves from D to B in Fig. 3, player P traverses a fairly short distance. The limiting motion of the players along a singular path at $\nu = \infty$ is as follows: player P remains stationary, while player E moves but remains at a constant distance from P. On the flattened surface of the cone, as shown in Fig. 6, the path of E is an arc of a circle. The length of the segment OE' in Fig. 6 is $\rho_B{}^*$, and that of OP is one, i.e. the position PE' is in the set B. Since $\rho_B{}^* = \cos\alpha$, it follows that PE' has a length of $\sin\alpha$ and is perpendicular to OE', while the circular arc EE' touches the segment OE' at E'. The point E defines the initial position of the evader and the length of the segment OE is equal to the limiting value $\rho_0{}^* = 1 - \sin\alpha$.

Note that as $\nu \to \infty$ the game may be compared with the following limiting optimal control problem for players E, with player P remaining stationary. The object of player E is to reach some point on the cone sufficiently far from the apex compared with the distance OP. Player E may move along any continuous path, obeying any law of motion; in so doing he must try to maximize the minimum distance from P.

A solution of this limiting problem may be constructed by means of simple geometrical arguments. If player E is initially outside the curvilinear triangle OEE' of Fig. 6, then the optimum outcome is equal to the initial distance PE. The outcome for initial positions of E inside the triangle OEE' is independent of E—it is $\sin\alpha$. The path of player E must then pass through the point E' without penetrating the tangent circle.

Figure 7 illustrates the level curves of the value of the game, in self-similar variables, for $\alpha = 2\operatorname{arctg}\frac{1}{2}$, $\nu = 2$ in the parameter domain Π_2 (these curves are symmetric about the horizontal axis). Since the value of the game for the domain Z_1 is equal to the initial distance between the players, the level curves will be circles $1 + \rho^2 - 2\cos\varphi = \text{const}$ about the point $\rho = 1$, $\varphi = 0$. In Z_2 the level curves must be plotted numerically. Knowing the level curves of the value of the game, we can approximate optimal controls for both players—i.e. construct an optimal synthesis—at each point of the phase space.

5. THE CASE $\nu = 1$

The approach problem for $\nu = 1$ is interesting as it is intermediate between pursuit and approach problems in general and combines some of their properties.

When $\nu = 1$ the sets B_0 and B are identical, $\rho_B = 1$, and all of Γ_1 is a dispersal surface. A primary path $-\Gamma$, (a circle) exists which touches the ray $\varphi = \alpha$ at ρ_B. Computations show that the singular curve BD approaches this path as $\nu \to 1+0$. The algorithm proposed in Sec. 3 for constructing the singular surface

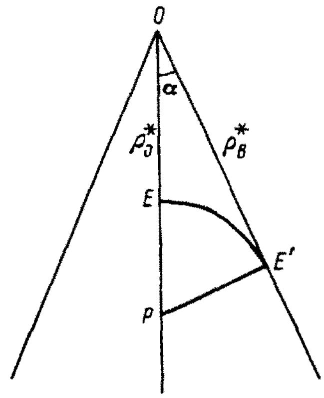

FIG. 6.

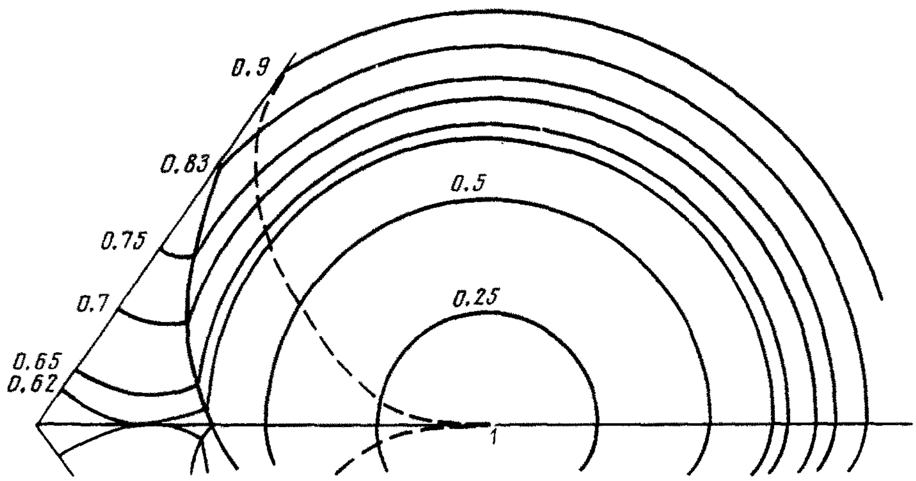

FIG. 7.

is useless when $\nu=1$, because Eqs (3.3) and (3.6) become meaningless at the points of B.

To complete the synthesis pattern, that is, to fill out the curvilinear triangle BOD ($B=(1, \alpha)$, $D=(1, 0)$), with optimal secondary paths, we will make the following assumption. The path Γ_* is the boundary of the primary domain and all secondary paths touch it at B, i.e. they form an integral funnel of paths "flowing" out of the secondary domain at B. The gradient p of the value of the game is discontinuous; it cannot be extended continuously from the secondary domain to B. The one-parameter family (3.8) of limit values of the gradient of the function $V(x_1, x_2, y_1, y_2)$ at $z_B=(1, 0, 1, 0)$ has the form

$$p(\lambda) = \frac{1}{\sqrt{a^2+1}}(a, \lambda, a, \lambda), \quad |\lambda| \leqslant 1, \quad a = \mathrm{tg}\,\alpha/2$$
$$p(0) = p_B = {}^1/_2\,(q^+ + q^-), \quad p(\pm 1) = q^\pm \tag{5.1}$$

A one-parameter family of regular paths $z(\tau, \lambda)=z_B - F_p(p(\lambda))\tau$, $\tau \geqslant 0$ emanates from z_B in retrograde time. By (5.1), the path $z(\tau, \pm 1)$ coincides with a tangent primary path—the circle Γ_* in the ρ, φ plane, while the function $z(\tau, 0)$ in ρ, φ variables corresponds to motion along the segment OB in Fig. 3.

The intermediate paths $z(\tau, \lambda)$ where $0<\lambda<1$, form an integral funnel and fill out the triangle OBD (Fig. 3).

As observed previously, all regular paths in polar coordinates ρ and φ are circles. In this case ($\nu=1$), all the secondary regular paths pass through the point B, where they touch the straight line OB; the latter is a limit element of the family—a circle of infinite radius. The centres of the circles in the family lie on the straight line perpendicular to OB through B. The formula for the radius is $R=a/\lambda$, $a=\mathrm{tg}\alpha/2$. When

$\lambda = \pm 1$ we obtain the boundary primary paths (the circle Γ_*); the value $\lambda = 0$ corresponds to the straight-line path OB.

The constructions of Secs 4 and 5 rely on the necessary conditions (2.1)–(2.3) and (3.1)–(3.7) for optimality. The question of whether these conditions are also sufficient requires further investigation.

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